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HYDRODYNAMICS OF FLOW IN AN ELASTIC PIPELINE

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Certain general propositions of the hydrodynamics of flow in an elastic pipeline are considered. The momentum equation for the flow under study has been derived. The fundamental possibility of solving the equations of hydrodynamics in an elastic pipeline in the form of solitary traveling waves has been shown. The influence of the viscosity of the liquid on the pulsed regime of flow has been investigated.

The hydrodynamics of flow in elastic pipelines of systems into which a certain volume of a liquid is ejected has its own features, which should be taken into account in designing such pipelines. We consider first the general regularities of propagation of a pressure wave in an elastic pipeline. We employ the approximation of a thin-walled elastic tube.

Let a pressure wave propagate in the pipeline; for the sake of simplicity we will assume that the wave has the shape of a rectangle in longitudinal cross section (Fig. 1).

Since, according to Hooke's law, the increase in the cross-sectional area of the pipeline is in proportion to the excess pressure in it, we can assume that the plots of the pressure and the cross-sectional area are the same. In Fig. 1, they are shown as a single curve.

The arrival of the pressure wave at any site of the pipeline leads to a local expansion of its portion, i.e., to the appearance of a solitary wave. The volume of the portion of the pipeline of length l changes by

$$\Delta V = \Delta S l = \Delta S c t \, . \quad (1)$$

We consider the propagation of the solitary wave farther in the pipe. Let v_1 be the velocity of a liquid flowing into the region of the solitary wave (cross section 1) and v_2 be the velocity of a liquid flowing out of it (cross section 2). The decrease in the volume of the portion of the pipeline to the initial level due to the elastic compression of its walls can be written in the form

$$-\Delta V = \Delta Q t \, , \quad (2)$$

where $\Delta Q = S\Delta v$; $\Delta v = v_2 - v_1$. The minus sign points to the decrease in the volume ΔV. Equating the right-hand sides of formulas (1) and (2) and passing to the derivative, we find the velocity of the solitary pressure wave

$$c = -\frac{\partial Q}{\partial S} \, . \quad (3)$$

It is easy to show that the dependence of the cross-sectional area of the pipeline on the coordinate X and the time t satisfies the wave equation.

We write the equation of continuity of the liquid in the form [1]

$$\frac{\partial S}{\partial t} + \frac{\partial Q}{\partial X} = 0 \, . \quad (4)$$

Employing the rule of determination of a derivative function prescribed implicitly

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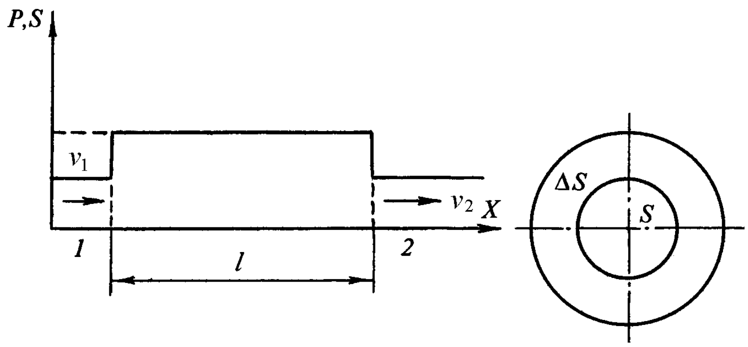

Fig. 1. Diagram for calculation of the velocity of propagation of a solitary wave in an elastic pipeline.

$$\frac{\partial Q}{\partial X} \Big/ \frac{\partial Q}{\partial S} = -\frac{\partial S}{\partial X},$$

we find

$$c = -\frac{\partial Q}{\partial S} = \frac{\partial Q}{\partial X} \Big/ \frac{\partial S}{\partial X}. \tag{5}$$

Consequently, Eq. (4) has the form

$$\frac{\partial S}{\partial t} + c\frac{\partial S}{\partial X} = 0. \tag{6}$$

Having established the time derivative of Eq. (6), we obtain the wave equation

$$\frac{\partial^2 S}{\partial t^2} = c^2\frac{\partial^2 S}{\partial X^2}. \tag{7}$$

It is well known that the velocity of motion of a liquid or a gas in a wave of any shape is related to the pressure and the velocity of its propagation:

$$v = \frac{P}{\rho c}, \tag{8}$$

where ρc is the hydraulic analog of acoustic resistance.

Multiplying both sides of equality (8) by the cross-sectional area of the pipeline S and finding the derivative of the quantity PS with respect to the flow rate Q, we obtain

$$\rho c = \frac{\partial\,(PS)}{\partial Q}. \tag{9}$$

Having noted that $\frac{\partial(PS)}{\partial S} \Big/ \frac{\partial(PS)}{\partial Q} = -\frac{\partial Q}{\partial S}$, we transform formula (9) with account for (3)

$$\rho c = \frac{\partial\,(PS)}{\partial Q} = \frac{\partial\,(PS)}{c\partial S}. \tag{10}$$

Consequently, the velocity of the pressure wave is equal to

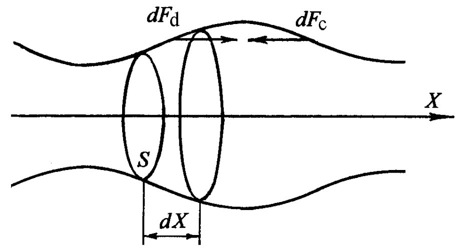

Fig. 2. Forces acting on the liquid in the region of the solitary wave on the source side of the walls of the elastic pipeline.

$$c = \sqrt{\frac{\partial (PS)}{\rho \partial S}} \, . \tag{11}$$

Taking into account [2] that $c = \sqrt{D/\rho}$, we find

$$D = \frac{\partial (PS)}{\partial S} \, . \tag{12}$$

Formula (12) determines the relationship between pressure in the liquid and the cross-sectional area of the pipeline.

We derive the momentum equation for the liquid flow in question. The change in the cross-sectional area of the elastic pipeline in the wave consists of two parts: divergent (diffuser) and convergent (confuser) parts (Fig. 2).

The second Newton law in projection to the X axis for the volume element of the liquid in the tube $dV = SdX$ can be written as

$$dF_{\text{d}} + N - (N + dN) - dF_{\text{c}} = \rho dV \frac{dv}{dt} \, , \tag{13}$$

where $dF_{\text{d}} = dF_{\text{c}} = PdS$ are the moduli of longitudinal components of forces acting on the volume element dV on the source side of the diffuser and confuser parts of the tube walls (they are equal according to the third Newton law) and $N = PS$ is the longitudinal component of pressure forces, which ensures flow of the liquid.

The force F_{d} acting on the volume element dV over the entire interior surface of the diffuser part is counterbalanced by the force F_{c} acting on the source side of the confuser part. If $dF_{\text{c}} = 0$, i.e., the diffuser is open, Eq. (13) is transformed as follows:

$$-\frac{dP}{dX} = \rho \frac{dv}{dt} \, . \tag{14}$$

The momentum equation for S = const has the same form. However in the case of simultaneous existence of the diffuser and confuser parts of symmetric geometry the momentum equation (13) is transformed to the form

$$-\frac{\partial (PS)}{S \partial X} = \rho \frac{dv}{dt} \, . \tag{15}$$

Consequently, in the case of a tube of constant cross section or divergent and convergent sections we can use the Euler equation (14). If the divergence of the tube is followed by convergence, we must employ Eq. (15). Such a form of the equation is convenient, for example, in solving the Zhukovskii problem on hydraulic shock in an elastic pipeline [3].

With account for

$$\frac{\partial (PS)}{\partial X} \Big/ \frac{\partial (PS)}{\partial S} = -\frac{\partial S}{\partial X} \tag{16}$$

and (12), we find

$$c^2 \frac{\partial S}{S\partial X} = \frac{dv}{dt} . \tag{17}$$

The system of equations (4) and (15) has been solved in [2] by passage to the nonlinear Schrödinger equation with logarithmic nonlinearity.

Let us prove that the system of equations (4) and (17) is of parabolic type and has solutions in the form of a traveling wave. We multiply Eq. (4) by the velocity v and add it to Eq. (17) multiplied by S. We employ (17) in the form

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial X} = c^2 \frac{\partial S}{S\partial X} . \tag{18}$$

As a result we find

$$\frac{\partial Q}{\partial t} + \frac{\partial (Qv)}{\partial X} = c^2 \frac{\partial S}{\partial X} . \tag{19}$$

Differentiating (19) with respect to the time t, we obtain

$$\frac{\partial^2 Q}{\partial t^2} + \frac{\partial^2 (Qv)}{\partial X \partial t} = c^2 \frac{\partial^2 S}{\partial X \partial t} . \tag{20}$$

From (4) we find

$$\frac{\partial^2 S}{\partial t \partial X} = - \frac{\partial^2 Q}{\partial X^2} . \tag{21}$$

Having substituted (21) into (20), we obtain

$$\frac{\partial^2 Q}{\partial t^2} + \frac{\partial^2 (Qv)}{\partial X \partial t} + c^2 \frac{\partial^2 Q}{\partial X^2} = 0 . \tag{22}$$

The solution of Eq. (22) will be sought in the form of the traveling wave $Q = Q(\zeta)$, where $\zeta = kX - \omega t$. From (22) we have

$$(\omega^2 + c^2 k^2) \frac{\partial^2 Q}{\partial \zeta^2} = \omega k \frac{\partial^2 (Qv)}{\partial \zeta^2} . \tag{23}$$

Denoting the positive quantity by $\gamma = \omega k/(\omega^2 + c^2 k^2)$, we find the relationship between Q and v in the form

$$Q (1 - \gamma v) = A\zeta + B . \tag{24}$$

Determining the velocity v from (24) and substituting it into (22), we obtain

$$\frac{\partial^2 Q}{\partial t^2} + \frac{1}{\gamma} \frac{\partial^2 Q}{\partial X \partial t} + c^2 \frac{\partial^2 Q}{\partial X^2} = 0 . \tag{25}$$

Relation (25) is an equation of canonical type with constant coefficients. The determinant of the type of equation [4] is equal to

$$a_{12}^2 - a_{11}a_{22} = \left(\frac{1}{2\gamma}\right)^2 - c^2 = \left(\frac{\omega^2 + c^2k^2}{2\omega k}\right)^2 - c^2 = \left(\frac{\omega^2 - c^2k^2}{2\omega k}\right)^2 . \quad (26)$$

Taking into account that $\omega = ck$, we have $a_{12}^2 - a_{11}a_{22} = 0$; consequently, the system of equations (4) and (17) is of parabolic type.

Up to this point we considered flow of an ideal liquid in an elastic tube. In analyzing the momentum equation for a viscous liquid, we assume that the amplitude of solitary waves in such a tube is small as compared to the length. In this case, in accordance with [1], we can disregard nonlinear convective terms in the momentum equation (15).

The momentum equation (average in accordance with the parabolic velocity profile) for flow in an elastic pipe with allowance for viscosity with small changes in the cross-sectional area S has the form [5]

$$\rho S \frac{\partial v}{\partial t} + \frac{\partial PS}{\partial X} = -\frac{8\pi\mu}{S} Q , \quad (27)$$

where $\mu = \rho\nu$. In this equation, as has already been noted, we have disregarded the nonlinear convective term $\rho S v \frac{\partial v}{\partial X}$.

Multiplying (4) by ρv and adding it to (27), we obtain the transfer equation for Q

$$\rho S \frac{\partial Q}{\partial t} + \rho Q \frac{\partial Q}{\partial X} + S \frac{\partial PS}{\partial X} + 8\pi\mu Q = 0 . \quad (28)$$

We consider small changes in the flow rate so that the nonlinear term $\rho Q \frac{\partial Q}{\partial X}$ can be disregarded. Consequently, we find

$$\rho \frac{\partial Q}{\partial t} + \frac{8\pi\mu}{S} Q = -\frac{\partial PS}{\partial X} . \quad (29)$$

Let us determine the similarity numbers of Eq. (29). We write the scales of the quantities

$$M_Q = \frac{Q}{Q^*} , \quad M_S = \frac{S}{S^*} , \quad M_t = \frac{t}{t^*} , \quad M_X = \frac{X}{X^*} , \quad M_P = \frac{P}{P^*} , \quad (30)$$

where the dimensionless quantities are asterisked. The dimensionless form of Eq. (29) has the form

$$\rho \frac{M_Q \partial Q^*}{M_t \partial t^*} + \frac{8\pi\mu}{M_S S^*} M_Q Q^* = -\frac{M_P M_S \partial P^* S^*}{M_X \partial X^*} . \quad (31)$$

We take the pressure scale $M_P = \rho \frac{M_Q^2}{M_S^2}$ in the form of a doubled dynamic head [6]. Carrying out the transformations, we obtain

$$\frac{\partial Q^*}{\partial t^*} + \frac{8\pi\nu}{M_S S^*} M_t Q^* = -\frac{M_Q M_t}{M_S M_X} \frac{\partial P^* S^*}{\partial X^*} . \quad (32)$$

The similarity of motion of the liquid in the nonstationary case is determined by the equality of two numbers: the Strouhal number and the Reynolds number [1]. The scale combination on the right-hand side of (32) represents the Strouhal number $\mathrm{Sh} = M_Q M_t/(M_S M_X)$. Consequently, the combination $M_S/\nu M_t$ is the Reynolds number for the elastic pipeline. Assuming that the time scale is a quantity determined by the period T of repetition of pressure pulses [1], we

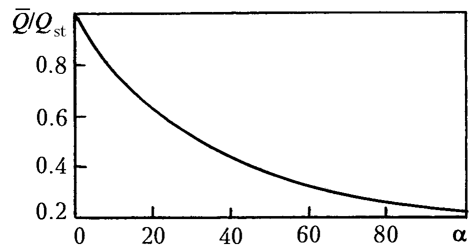

Fig. 3. Average rate of flow of the liquid through the elastic pipeline vs. dimensionless parameter α.

take $M_t = T = 2\pi/\omega$, where ω is the cyclic frequency of repetition of solitary waves in this case. Setting $M_S = \pi r^2$, we find, accurate to a constant coefficient, the analog of the Reynolds number for pulsed flow in the elastic pipeline $\alpha = \omega r^2/\nu$. The dimensionless parameter α found primarily characterizes the frequency of repetition of solitary waves in the pipeline.

The solution of Eq. (29) with the conditions $\alpha = 0$ and $Q = Q_{st}$ and $\alpha = \infty$ and $Q = 0$ has the form

$$\frac{Q}{Q_{st}} = \left[1 - \exp\left(-\frac{8\omega t}{\alpha}\right)\right], \tag{33}$$

where $Q_{st} = -\frac{S}{8\pi\mu}\frac{\partial PS}{\partial X}$ is the flow rate in the stationary case, found from (29) with the condition $\frac{\partial Q}{\partial t} = 0$. Thus, the instantaneous flow rate Q lags behind the pressure gradient applied and is not determined just by the instantaneous pressure gradient.

We find the average value of the flow rate of the liquid over the period of repetition of pulses T

$$\overline{Q} = \frac{1}{T}\int_0^T Q dt = Q_{st}\left[1 + \frac{\alpha}{16\pi}\left(\exp\left(-\frac{16\pi}{\alpha}\right) - 1\right)\right]. \tag{34}$$

Figure 3 shows the relative (average over the period) change in the flow rate of the liquid in the elastic tube as a function of the parameter α. It is seen from the plot that the average rate of flow through the elastic tube decreases with increase in the frequency of repetition of solitary waves. The reason is that, as the pulse repetition frequency increases, the unformed previous solitary wave hinders the traversal of the wave that follows.

NOTATION

A and B, integration constants; $a_{i,j}$, coefficients of the differential equation; c, velocity of the pressure wave; D, elasticity of the walls of the elastic tube; dF_d and dF_c, differentials of the moduli of longitudinal components of forces acting on the volume element dV on the source side of the diffuser and confuser parts of the tube walls; dV, differential (element) of the volume of the elastic tube; k, wave number; l, length of the pipeline's portion; M_i, scales of the quantities; N, longitudinal component of pressure forces; P, pressure in the wave; Q, instantaneous value of the flow rate of the liquid; Q_{st}, rate of flow in the pipe in the stationary case; $\overline{Q}$, average value of the flow rate of the liquid over the period of repetition of pulses; r, radius of the elastic pipeline when the external and internal pressures are equal; S, cross-sectional area of the pipeline; Sh, Strouhal number; T, period of repetition of pressure pulses; t, time of propagation of a pressure wave to the distance l; v, longitudinal velocity of the liquid; v_1, velocity of the liquid flowing into the region of the solitary wave; v_2, velocity of the liquid flowing out of it; X, longitudinal coordinate; α, dimensionless parameter, analog of the Reynolds number; γ, number equal to $\omega k/(\omega^2 + c^2k^2)$; ΔQ, difference of the flow rates of the liquid in cross sections 1 and 2 due to the compression of the pipeline; ΔS, increase in the cross-sectional area in the solitary wave; ΔV, change in the volume of the pipeline portion of length l; ζ, wave phase; μ, dynamic viscosity of the liquid; ν, its kinematic viscosity; ρ, density of the liquid; ω, cyclic frequency of the wave.

Subscripts and superscripts: d, diffuser; c, confuser; st, stationary; 1 and 2, initial and final cross sections of the pipeline in the solitary wave; *, dimensionless quantity; P, Q, S, t, and X characterize the scale of the quantity.

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